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A Simple Relation between Non-locality and a Corrected Entropy

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Abstract Within the framework of a statistical interpretation of quantum mechanics entanglement (in a mathematical sense) manifests itself in the non-separability of the statistical operator ρ representing the ensemble in question. In experiments, on the other hand, entanglement can be detected, in the form of non-locality, by the violation of Bell's inequality $\Delta \leq 2$. How do these different viewpoints match? We employ a corrected von Neumann entropy to measure the (mathematical) degree of entanglement and show that, at least in the case of 2×2 dimensions, this function is directly related to Bell's correlation function Δ. This relation can be well approximated by an ellipse equation which, for the first time, allows for a direct comparison of the two faces of entanglement.

Keywords Entanglement measure · Locality · Entropy

1 Introduction

The physical principle underlying the notion of entanglement has been detected by Schrödinger [1] in his first analysis of the Einstein-Podolsky-Rosen (EPR) problem [2]. Meanwhile, the most fascinating technological implications of quantum mechanics (QM) are based on said notion: Entanglement is the essential ingredient for both quantum cryptography, quantum computing, and quantum teleportation [3]. A significant problem, however, is: What in fact *is* entanglement? We will address this question in connection with statistical operators which allow for a more general formulation of QM. If we accept that a statistical

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operator represents an ensemble,[1] then entanglement must be reflected by a mathematical property of said operator which quite naturally is its non-separability. In this contribution we will *not* refer to the different possible definitions of separability which have been compared and discussed previously [4]. Instead we will look first for a *general* measure of (mathematical) entanglement which is independent of the separability definition in use.

One could argue that the so-called entanglement of formation (quantifying "the resources needed to *create* a given entangled state" [5]), the "concurrence", and entanglement witnesses [6] are already meaningful measures of (which?) entanglement. However, all these measures refer to a completely operationalistic interpretation of QM and therefore are of *very* limited use only.

In the sense of Einstein, and based on the work of Bell and many others, we may say that the relation between two completely separated parts of a former common whole is *local* if (and only if) Bell's inequality $\Delta \leq 2$ holds, where

$$\Delta := |O(\vec{a}, \vec{b}) - O(\vec{a}, \vec{b}')| + |O(\vec{a}', \vec{b}) + O(\vec{a}', \vec{b}')| \tag{1}$$

and $O(\vec{a}, \vec{b})$ being the output of an appropriate experiment where the actual setting of the joint apparatus is given by the two vectors $\vec{a}$ and $\vec{b}$ (see below). One may ask how locality and the (mathematical) entanglement described above are related. It is obvious that a precisely definable relation must exist, because the above mentioned outputs in QM are given by

$$O(\vec{a}, \vec{b}) = \mathrm{Tr}(\hat{P}(\vec{a}, \vec{b})\rho) \tag{2}$$

with $\hat{P}$ representing the measurement apparatus, and, on the other hand, the measure of (mathematical) entanglement must also be a direct function of ρ. To find the mapping of one of these properties onto the other will be the topic of this contribution and it will be shown that the claim of Hiesmayr [7] that "nonlocality and entanglement are indeed two different features" is wrong.

2 Definition of the Model

Assume that a source produces pairs of entities (A_i, B_i) which dissociate after being generated. The A_i are sent to an observer named Alice while the B_i are sent to her colleague Bob. Each of them measures a rotationally variant property type as, e.g., spin or polarization on A_i and B_i, respectively. Let the measurement apparatus A (B) be represented by the self-adjoint operator $\hat{A}$ ($\hat{B}$). We assume that the eigenvalues of both operators are ± 1, and that $\mathcal{H}$ is the 2×2-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ with an orthonormal basis $\{|\alpha_i\rangle|\beta_i\rangle\}$. We further assume that the orientation of A with respect to the laboratory coordinate system is given by the vector $\vec{a}$, and that

$$\hat{A} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

[1] Here and in the following it does not matter whether we consider an ontic or an effective ensemble, i.e., it does not play any role whether the ensemble in question consists of a lot of identical but non-interacting particles/objects/entities or whether it consists of a series of events generated, e.g., as the effect of a repeated measurement on one particle/object/entity.

Now let apparatus B differ from A insofar as the vector $\vec{b}$ determining its actual internal status can be obtained from $\vec{a}$ by a rotation around an axis perpendicular to it. Then $\hat{B}$ emerges from $\hat{A}$ by a rotation around the angle χ between $\vec{a}$ and $\vec{b}$.

$$\Rightarrow \quad \hat{B} = \begin{pmatrix} \cos\chi & \sin\chi \\ \sin\chi & -\cos\chi \end{pmatrix}. \tag{4}$$

In complete analogy we define two other operators, $\hat{A}'$ and $\hat{B}'$, where $\hat{A}'$ represents apparatus A rotated with respect to its first position (determined by $\vec{a}$) by an angle φ. $\hat{B}'$ stands for B rotated with respect to $\vec{a}$ by an angle ψ. Note that $\hat{A}$ and $\hat{A}'$ as well as $\hat{B}$ and $\hat{B}'$ are in general non-commuting. The determinants of the commutators attain their maximum if $\varphi = \frac{\pi}{2}$ and $\psi = \chi + \frac{\pi}{2}$, respectively, and this is the angle setting choosen in the following analysis. We furthermore fix χ to $\pi/4$, because with this value the maximal violation of Bell's inequality by ρ_{sing} (see below) can be achieved [8].

With these four apparatus adjustments we can perform four experiments, i.e., four large series of single runs. Each single run in one series yields an outcome $O_i = A_i \cdot B_i$. The result of each experiment is the average over all single runs which, in terms of QM, is given by

$$O(\vec{a}, \vec{b}) = \mathrm{Tr}\left((\hat{A} \otimes \hat{B})\rho\right). \tag{5}$$

Finally we calculate the correlation function Δ according to (1).

This procedure is carried out with different statistical operators on $\mathcal{H}$ obeying the general formula

$$\rho = \varepsilon\rho_{sing} + (1-\varepsilon)\rho_i \tag{6}$$

where

$$\rho_{sing} = \frac{1}{2}(\hat{A}_{11} \otimes \hat{B}_{22} - \hat{A}_{12} \otimes \hat{B}_{21} - \hat{A}_{21} \otimes \hat{B}_{12} + \hat{A}_{22} \otimes \hat{B}_{11}) \tag{7}$$

represents the famous singlet state of two coupled electron spins where $\hat{A}_{11} = |\alpha_1\rangle\langle\alpha_1|$ and the other operators defined analogously. This operator is maximally non-separable with respect to all definitions of separability. The ρ_i are chosen to represent the possible limiting cases in this Hilbert space.

The ρ with $0 < \varepsilon < 1$ reflect the realistic situation in EPR experiments that the ensembles actually used are not completely non-separable, as desired, i.e., their statistical operator ρ_{sing} is contaminated by a separable part ρ_i.

3 A General Measure of Mathematical Entanglement

The von Neumann entropy

$$S = -\mathrm{Tr}(\rho \ln \rho) \tag{8}$$

provides a measure of mathematical entanglement, *because* if those parts of an operator are omitted which take care of its non-separability, also certain connections are lost, and in consequence the entropy must increase. This measure, however, is only *relative* insofar as the entropy is influenced by the mixedness of an operator as well.[2] We therefore suggest a correction to S by a measure of the degree of mixedness, and since a statistical operator ρ is said to be mixed if (and only if) $\operatorname{Tr}\rho^2 < \operatorname{Tr}\rho = 1$, we define the corrected von Neumann entropy as

$$S_{corr} = S - (1 - \operatorname{Tr}\rho^2). \tag{9}$$

If ρ represents a pure state the correction term vanishes. If, on the other hand, the state represented by ρ is maximally mixed, i.e., if, on the 2×2 dimensional Hilbert space, $\rho = 1/4(\hat{A}_{11} \otimes \hat{B}_{11} + \hat{A}_{11} \otimes \hat{B}_{22} + \hat{A}_{22} \otimes \hat{B}_{11} + \hat{A}_{22} \otimes \hat{B}_{22})$, then the correction term amounts to 3/4. In the first case we obtain $S_{corr} = 0$, whereas in the second case $S_{corr} = 2\ln 2 - 3/4 \approx 0.6363$.

S_{corr} is asymptotically correct which means that for $\varepsilon = 1$ we obtain $S_{corr} = 0$, and $\varepsilon = 0$ yields $S_{corr} = -\operatorname{Tr}(\rho_i \ln \rho_i) - 1 + \operatorname{Tr}\rho_i^2$ which is what one would expect from (9). Moreover, the corrected entropy is subadditive and satisfies the Araki-Lieb inequality. The proof is given in Appendix A. Note that, although the set of all ρ is convex, the corrected entropy is not. This, however, does not seem to be a relevant obstacle, because also the von Neumann entropy itself is not convex.

4 Results

The first separable operator to investigate is the above-mentioned

$$\rho_1 = \frac{1}{4}(\hat{A}_{11} \otimes \hat{B}_{11} + \hat{A}_{11} \otimes \hat{B}_{22} + \hat{A}_{22} \otimes \hat{B}_{11} + \hat{A}_{22} \otimes \hat{B}_{22}). \tag{10}$$

It can be decomposed into $\rho_A \otimes \rho_B$ and represents a maximally mixed state. The density matrix of the corresponding ρ (see (6)) is then given by

$$M_1 = \frac{1}{4}\begin{pmatrix} 1-\varepsilon & 0 & 0 & 0 \\ 0 & 1+\varepsilon & -2\varepsilon & 0 \\ 0 & -2\varepsilon & 1+\varepsilon & 0 \\ 0 & 0 & 0 & 1-\varepsilon \end{pmatrix}. \tag{11}$$

From this we obtain the corrected entropy

$$S_{corr,1}(\varepsilon) = -\frac{3(1-\varepsilon)}{4}\ln\left(\frac{1-\varepsilon}{4}\right) - \frac{1+3\varepsilon}{4}\ln\left(\frac{1+3\varepsilon}{4}\right) - \frac{3}{4}(1-\varepsilon^2) \tag{12}$$

and the correlation function

$$\Delta_1(\varepsilon) = 2\sqrt{2}\varepsilon. \tag{13}$$

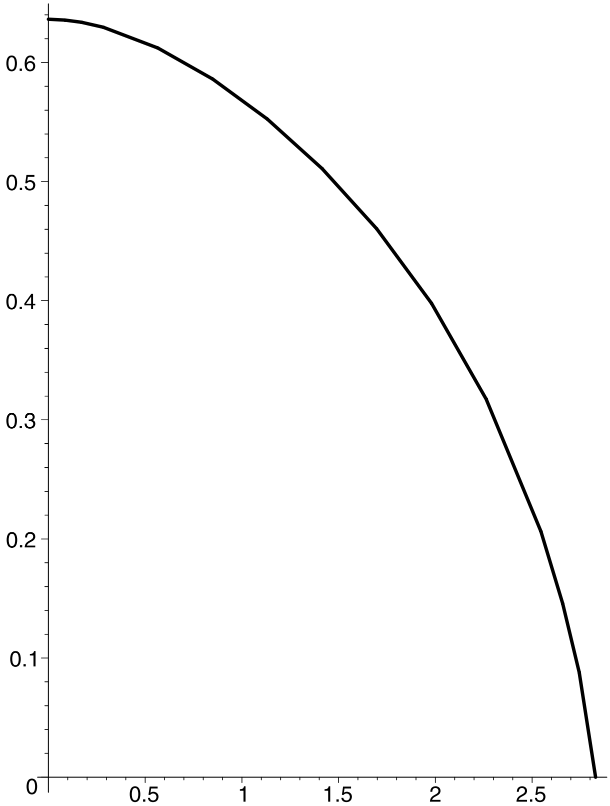
Figure 1 shows the relation between these two quantities.

[2] But note that S is *not* a measure of mixedness which is given by $1 - \operatorname{Tr}\rho^2$, because, if $\ln\rho$ is expanded into a power series, we obtain

$$S = -\operatorname{Tr}\left(\sum_{i=1}^{\infty} a_i \rho^i\right)$$

which is *always* different from $1 - \operatorname{Tr}\rho^2$.

Fig. 1 $S_{corr,1}$ (y-axis) vs. Δ_1 (x-axis)



A simple analogue to ρ_1 which, however, can *not* be written as the direct product of a ρ_A and a ρ_B is given by

$$\rho_2 = \frac{1}{2}(\hat{A}_{11} \otimes \hat{B}_{11} + \hat{A}_{22} \otimes \hat{B}_{22}), \tag{14}$$

yielding the density matrix

$$M_2 = \frac{1}{2}\begin{pmatrix} 1-\varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & -\varepsilon & 0 \\ 0 & -\varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 & 1-\varepsilon \end{pmatrix} \tag{15}$$

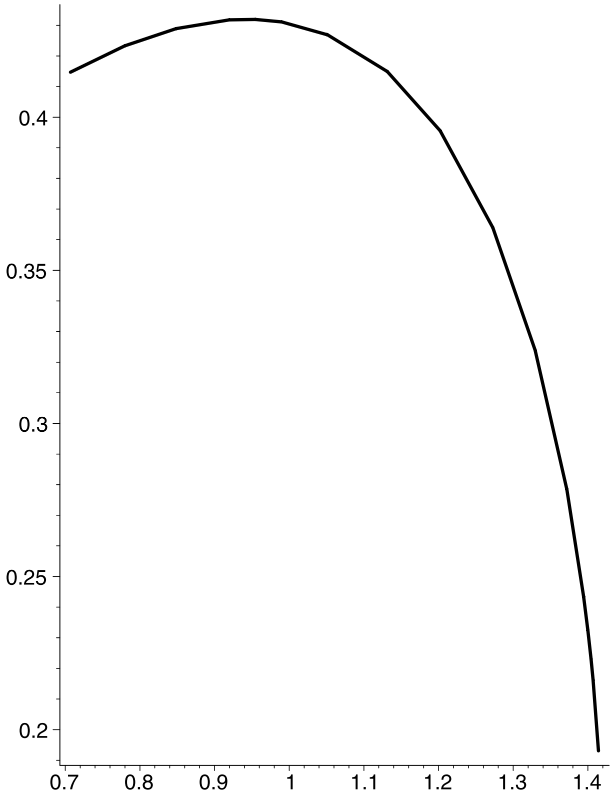
and the corrected entropy

$$S_{corr,2}(\varepsilon) = -\varepsilon \ln \varepsilon - (1-\varepsilon)\ln\left(\frac{1-\varepsilon}{2}\right) - \varepsilon - \frac{1-3\varepsilon^2}{2}. \tag{16}$$

We now have to perform a case differentiation, because the function

$$\Delta_2(\varepsilon) = \sqrt{2}(|1-2\varepsilon| + \varepsilon) \tag{17}$$

Fig. 2a $S_{corr,2}$ (y-axis) vs. Δ_2 (x-axis), first branch



consists of two different branches which meet at $\varepsilon = 1/2$, but at this point Δ_2 is not differentiable. For $\varepsilon \in [1/2, 0]$ the relation between $S_{corr,2}$ and Δ_2 is shown in Fig. 2a, for $\varepsilon \in [1/2, 1]$ in Fig. 2b.

The operators investigated so far represent mixed states. In contrast we now use

$$\rho_3 = \frac{1}{2}(\hat{A}_{11} \otimes \hat{B}_{11} + \hat{A}_{12} \otimes \hat{B}_{12} + \hat{A}_{21} \otimes \hat{B}_{21} + \hat{A}_{22} \otimes \hat{B}_{22}) \tag{18}$$

to contaminate ρ_{sing}. It represents a pure state and can not be decomposed into $\rho_A \otimes \rho_B$. From the corresponding density matrix

$$M_3 = \frac{1}{2}\begin{pmatrix} 1-\varepsilon & 0 & 0 & 1-\varepsilon \\ 0 & \varepsilon & -\varepsilon & 0 \\ 0 & -\varepsilon & \varepsilon & 0 \\ 1-\varepsilon & 0 & 0 & 1-\varepsilon \end{pmatrix} \tag{19}$$

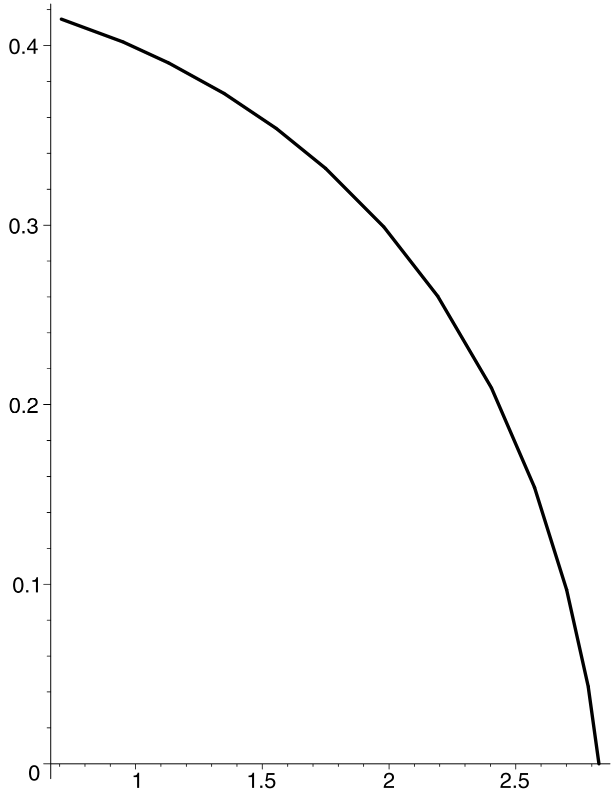
we obtain

$$S_{corr,3}(\varepsilon) = -\varepsilon \ln \varepsilon - (1-\varepsilon)\ln(1-\varepsilon) - 2\varepsilon(1-\varepsilon) \tag{20}$$

and

$$\Delta_3(\varepsilon) = 2\sqrt{2}|1-2\varepsilon|. \tag{21}$$

Fig. 2b $S_{corr,2}$ (y-axis) vs. Δ_2 (x-axis), second branch



Though also this correlation function consists of two branches, a case differentiation is not necessary, because the branches are symmetric with respect to $\varepsilon = 1/2$. The corrected entropy can be found as a function of Δ in Fig. 3.

With the final operator

$$\rho_4 = \frac{1}{4}(\hat{A}_{11} + \hat{A}_{22}) \otimes (\hat{B}_{11} + \hat{B}_{12} + \hat{B}_{21} + \hat{B}_{22}) \tag{22}$$

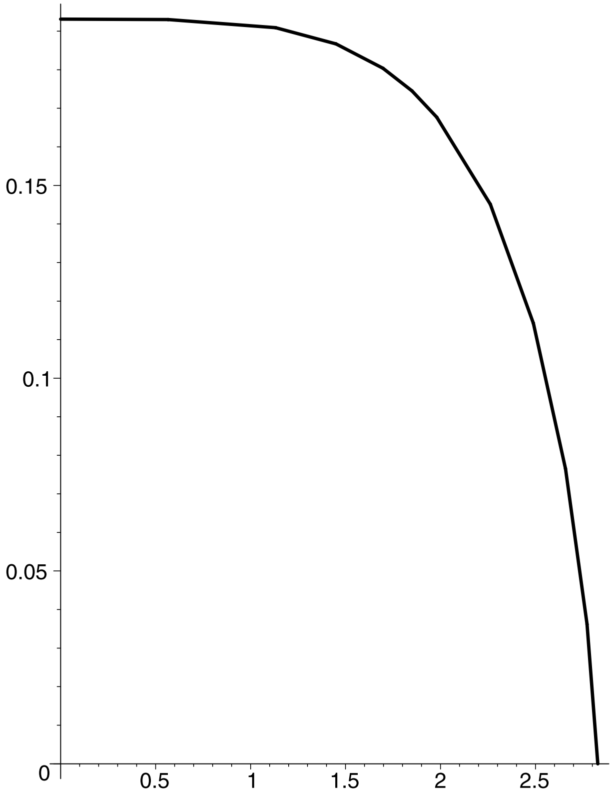
we have a decomposable operator which consists of one pure and one maximally mixed part. The density matrix

$$M_4 = \frac{1}{4}\begin{pmatrix} 1-\varepsilon & 1-\varepsilon & 0 & 0 \\ 1-\varepsilon & 1+\varepsilon & -2\varepsilon & 0 \\ 0 & -2\varepsilon & 1+\varepsilon & 1-\varepsilon \\ 0 & 0 & 1-\varepsilon & 1-\varepsilon \end{pmatrix} \tag{23}$$

gives rise to a more complicated corrected entropy,

$$\begin{aligned} S_{corr,4}(\varepsilon) = &-\frac{1-\varepsilon}{2}\ln\frac{1-\varepsilon}{2} - \frac{1+\varepsilon+z}{4}\ln\frac{1+\varepsilon+z}{4} \\ &- \frac{1+\varepsilon-z}{4}\ln\frac{1+\varepsilon-z}{4} - \frac{1+\varepsilon-2\varepsilon^2}{2}, \end{aligned} \tag{24}$$

Fig. 3 $S_{corr,3}$ (y-axis) vs. Δ_3 (x-axis)



with $z = \sqrt{5\varepsilon^2 - 2\varepsilon + 1}$, whereas

$$\Delta_4(\varepsilon) = 2\sqrt{2\varepsilon} \tag{25}$$

as in case 1, see (13). The functional relation between S_{corr} and Δ is shown in Fig. 4.

5 Interpretation

In the preceding section we have investigated the functional relation between the corrected entropy and Bell's correlation function by use of four different operators covering the possible range of contaminations of the operator representing the maximally entangled singlet state. In each case the implicit function $S_{corr}(\Delta)$ can be approximated by a segment of an ellipse obeying

$$\frac{(S_{corr} - S_{corr,\min})^2}{a^2} + \frac{(\Delta - \Delta_{\min})^2}{b^2} = 1. \tag{26}$$

Even a simple fit (using the maximum and end points of the curve to determine the parameters a and b, respectively), gives a reasonable accuracy of mostly more than 90% can be achieved. The values are given in Table 1.

But what will happen if we deviate from the ideal measurement conditions described in Sect. 2? It is a straightforward task to show that the ellipsoidal relation still holds.

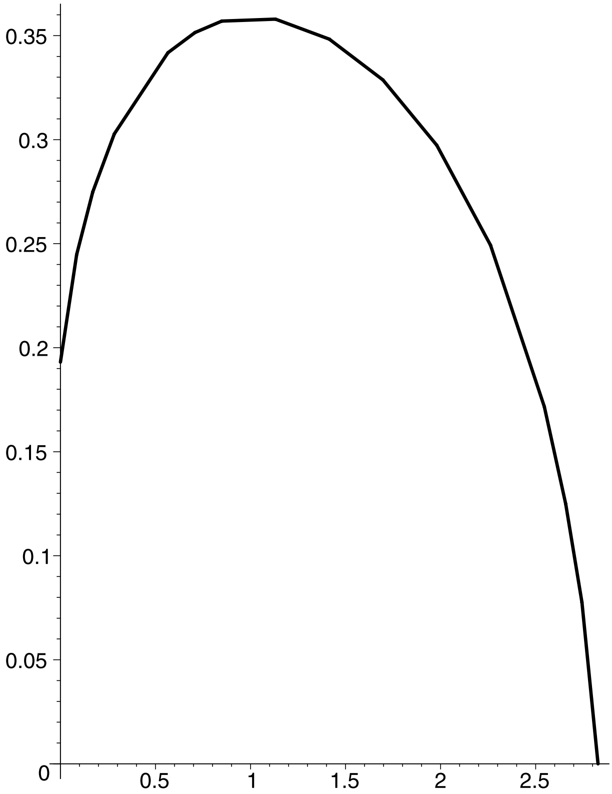


Fig. 4 $S_{corr,4}$ (y-axis) vs. Δ_4 (x-axis)

Table 1 Parameters of the ellipses ($e =$ numerical excentricity, p see (28))

Case no.	a^2	b^2	Deviation at half of the domain (%)	e	p	$\overline{S_{corr}}$
1	0.4049	8.0000	7.8	0.974	0.155	0.4621
2a	0.0570	0.2224	0.8	0.862	–	–
2b	0.1873	8.0000	2.0	0.988	0.250	0.2983
3	0.0373	8.0000	12.0	0.998	0.201	0.1667
4	0.1289	3.3056	1.9	0.980	0.326	0.2728

The data for the cases (i) $(\varphi, \psi, \chi) = (\pi/2, 5\pi/6, \pi/3$ (suboptimal angle χ) and (ii) $(\varphi, \psi, \chi) = (\pi, 5\pi/4, \pi/4)$ (commuting operators) are available from the author up request. In addition for $(\varphi, \psi, \chi) = (2\pi/3, \pi/3, \pi)$, which has been proposed by a referee, the ellipsoidal relation holds.

So far only operators depending on *one* parameter have been investigated. In Appendix B a first non-exotic example is treated where the statistical operator depends on *two* parameters. Also in this case an ellipsoidal relation between mathematical and physical entanglement can be found.

It is remarkable that, although at first glance the ellipses seem to be quite different, their numerical excentricity $e = \sqrt{b^2 - a^2}/b$ falls into a small interval around 0.960, i.e., the characteristic of the $S_{corr} - \Delta$ relation is nearly the same for *all* the various operators in question.

Moreover this e determines the circumference of the ellipses, because $U = 4bE(\pi/2, e)$ where

$$E\left(\frac{\pi}{2}, e\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2 \psi} d\psi \approx 1.087 \tag{27}$$

is the complete elliptic integral of the second kind, i.e., the differences in the circumference of the various ellipses are governed by the parameter b only.

The fraction

$$p = \frac{\int_2^{\Delta_{\max}} S_{corr}(\Delta) d\Delta}{\int_{\Delta_{\min}}^{\Delta_{\max}} S_{corr}(\Delta) d\Delta} \tag{28}$$

is the probability that Bell's inequality can be violated *irrespective* of the value of ε. Only the *type* of the contamination (not its actual extent) determines this probability, so knowledge of the contaminating operator allows a test of the viability of an EPR-type experiment. Interestingly it does not play any role if the insufficient operation of the EPR source is described by admixing either ρ_1 or ρ_2 to ρ_{sing}. Both the maximally mixed and the pure state give rise to low probabilities of 15 to 20%, i.e., if it is not possible to keep ε close to 1, the ensembles produced by the source are of no use. In contrast, p is highest for ρ_4, i.e., the chance to obtain $\Delta > 2$ is highest in this case.

Finally we can also define the mean corrected entropy via

$$\int_{\Delta_{\min}}^{\Delta_{\max}} S_{corr}(\Delta) d\Delta = \overline{S_{corr}} \cdot (\Delta_{\max} - \Delta_{\min}). \tag{29}$$

It is a measure for the reduction of the (mathematical) degree of entanglement by contaminating ρ_{sing} by a certain type of operator, again irrespective of the actual extent of the contamination. Table 1 shows that the effect of ρ_3 on ρ_{sing} is the least one. The mean corrected entropy is increased by 0.1667 only which is understandable because ρ_3 is the only operator representing a pure state as ρ_{sing}. In consequence, using a maximally mixed counterpart as ρ_1 leads to the highest entropy value.

6 Summary

Entanglement in a mathematical sense manifests itself in the non-separability of the statistical operator ρ. On the other hand, entanglement in a physical sense can be observed as the non-locality of the relation between two completely separated parts of a former common whole by the violation of Bell's inequality $\Delta \leq 2$.

In this contribution we have first shown that the von Neumann entropy is a reasonable measure of mathematical entanglement, *if* it is corrected by a term which takes into account the degree of mixedness of the operator in question, see (6). By use of a set of representative statistical operators for the 2×2 case it follows that this corrected entropy, S_{corr}, can be expressed as a function of Δ, that is, the two faces of entanglement are related by the elliptical equation $S_{corr}^2/a^2 + \Delta^2/b^2 = 1$.

All of the investigated ellipses have nearly the same numerical excentricity ϵ which demonstrates that the character of the $S_{corr} - \Delta$ relation is the same for all the various operators. Several other properties have been unveiled, and they corroborate the assumption that the ellipsoidal relation between mathematical and physical entanglement is an essential feature of the microworld.

Up to now this relation seems to be an empirical result only. However, it can be shown that it is the positive consequence of the comparison of the corrected entropy (essentially logarithmic with respect to the mixing parameter ε) with the correlation function where the linear dependence on ε is dominant (see Appendix C).

For a further analysis and interpretation of the ellipsoidal relation it could be of interest to refer to the Ermakov-Lewis invariant which, on the one hand, is of essentially elliptical character and, on the other hand, manifests the connection between quantum-mechanical and classical time evolution [9, 10].

Appendix A

Subadditivity is given if the entropy $S_{corr}(AB)$ of the composed system AB is bound from above by the sum of the entropies $S_{corr}(A)$ and $S_{corr}(B)$ of the two parts A and B, respectively, i.e., if

$$S_{corr}(AB) \le S_{corr}(A) + S_{corr}(B). \tag{A.1}$$

Inserting (9) we obtain

$$S(AB) - 1 + \operatorname{Tr}\rho^2 \le S(A) + S(B) - 2 + \operatorname{Tr}(\rho_A^2 + \rho_B^2). \tag{A.2}$$

Since the von Neumann entropy S is already subadditive, we only have to show that

$$1 + \operatorname{Tr}\rho^2 \le \operatorname{Tr}(\rho_A^2 + \rho_B^2) \tag{A.3}$$

in order to prove the subadditivity of S_{corr}.

Take ρ from (6) and let ρ_i be given in diagonal form:

$$\rho_i = \sum_{j,k} c_{jk}\hat{A}_{jj} \otimes \hat{B}_{kk}, \quad c_{jk} \ge 0\ \forall j,k \quad \text{and} \quad \sum_{j,k} c_{jk} = 1. \tag{A.4}$$

We then obtain

$$\operatorname{Tr}\rho^2 = \varepsilon^2 + \varepsilon(1-\varepsilon)(c_{12} + c_{21}) + (1-\varepsilon)^2(c_{11}^2 + c_{12}^2 + c_{21}^2 + c_{22}^2). \tag{A.5}$$

The statistical operator of system A is given by the partial trace of ρ over the degrees of freedom of system B yielding

$$\begin{aligned}\operatorname{Tr}\rho_A^2 &= \frac{\varepsilon^2}{2} + \varepsilon(1-\varepsilon)\\ &\quad + (1-\varepsilon)^2(c_{11}^2 + 2c_{11}c_{12} + c_{12}^2 + c_{21}^2 + 2c_{21}c_{22} + c_{22}^2).\end{aligned} \tag{A.6}$$

In the same way we obtain

$$\begin{aligned}\operatorname{Tr}\rho_B^2 &= \frac{\varepsilon^2}{2} + \varepsilon(1-\varepsilon)\\ &\quad + (1-\varepsilon)^2(c_{11}^2 + 2c_{11}c_{21} + c_{21}^2 + c_{12}^2 + 2c_{12}c_{22} + c_{22}^2).\end{aligned} \tag{A.7}$$

Inserting (A.7), (A.6), and (A.5) into (A.3) we end up with the condition

$$\begin{aligned} \varepsilon(1-\varepsilon)(c_{12}+c_{21})-1 \le \varepsilon^2 &+ 2\varepsilon(1-\varepsilon) \\ &+ (1-\varepsilon)^2(c_{11}^2+c_{12}^2+c_{21}^2+c_{22}^2) \\ &+ 2(1-\varepsilon)^2(c_{11}+c_{22})(c_{12}+c_{21}). \end{aligned} \tag{A.8}$$

It is easy to see that the left side of this inequality is always lower than or equal to $-3/4$. The evaluation of the right side, however, is a little bit more complicated. If ρ_i represents a pure state, then the right side reduces to

$$\varepsilon^2 + 2\varepsilon(1-\varepsilon) + (1-\varepsilon)^2, \tag{A.9}$$

because the condition

$$\sum_{j,k} c_{j,k}^2 = \sum_{j,k} c_{j,k} \tag{A.10}$$

can be fulfilled only if three of the coefficients are equal to 0, and it is seen immediately that (A.9) amounts to 1. If, on the other hand, ρ_i represents the maximally mixed state where all $c_{jk} = 1/4$, then the right side becomes equal to

$$-\frac{\varepsilon^2}{4} + \frac{\varepsilon}{2} + \frac{3}{4} \tag{A.11}$$

which is always larger than or equal to $3/4$.

Taking these results altogether we obtain the inequality

$$-\frac{3}{4} < +\frac{3}{4} \tag{A.12}$$

which is obviously correct.

The Araki-Lieb inequality

$$S_{corr}(AB) \ge |S_{corr}(A) - S_{corr}(B)| \tag{A.13}$$

holds as well. By use of the definition of the corrected entropy we obtain

$$S(AB) - 1 + \operatorname{Tr}\rho^2 \ge \left|S(A) - S(B) - \operatorname{Tr}(\rho_B^2 - \rho_A^2)\right| \tag{A.14}$$

which, employing the fact that $|a-b| \ge |a| - |b|$, boils down to

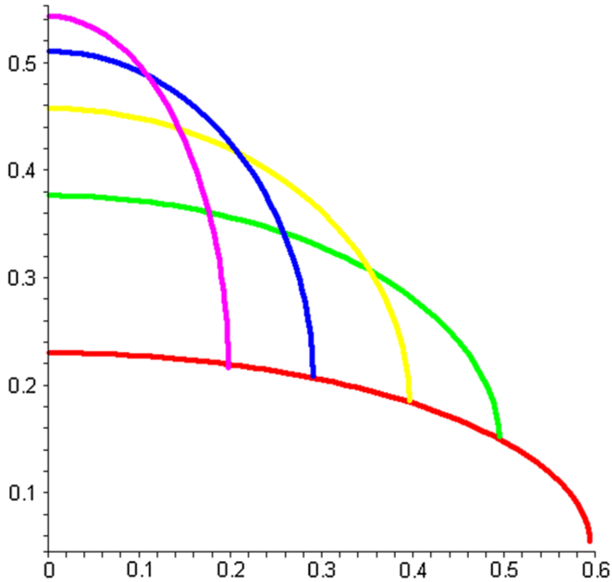
$$S(AB) - 1 + \operatorname{Tr}\rho^2 \ge |S(A) - S(B)| - \left|\operatorname{Tr}(\rho_B^2 - \rho_A^2)\right|. \tag{A.15}$$

For $\varepsilon = 1$ ρ coincides with ρ_{sing}, and (A.15) is met trivially. In the other extreme, $\varepsilon = 0$ so that $\rho = \rho_i$. If ρ_i is chosen to represent the maximally mixed state (see above), we obtain

$$\ln 4 - \frac{3}{4} \ge 0 \tag{A.16}$$

which is obviously true. Finally we use the fact that the dependence of both the von Neumann entropy and the trace over the squared operators is monotonous with respect to ε.

Fig. 5 (Color online) $S_{corr}(\Delta)$, $\alpha = 0.1$ (*pink*), $\alpha = 0.15$ (*blue*), $\alpha = 0.2$ (*yellow*), $\alpha = 0.25$ (*green*), and $\alpha = 0.3$ (*red curve*)



Appendix B

In the most general case the density matrix of a statistical operator on a 2×2-dimensional Hilbert space which is given by $\rho = \sum_{i,j,k,l=1}^{2} a_{ij} b_{kl} \hat{A}_{ij} \otimes \hat{B}_{kl}$ contains 8 free parameters (i) to satisfy $\mathrm{Tr}\,\rho = 1$ and (ii) to take care of the self-adjointness of ρ. However, these parameters may not be chosen at will, because all four eigenvalues of ρ have to belong to the interval $[0, 1]$ which means that there are additional restrictions to the range of $a_{11}, a_{22}, b_{11}, b_{22}, a_{12}$, and b_{12}. Now let us choose

$$\rho(\alpha,\beta) = \frac{1}{4}\begin{pmatrix} 1/2 & (1+i)\beta & (2+2i)\alpha & 2i\alpha\beta \\ (1-i)\beta & 1/2 & 2\alpha\beta & (2+2i)\alpha \\ (2-2i)\alpha & 2\alpha\beta & 3/8 & (3+3i)\beta \\ -2i\alpha\beta & (2-2i)\alpha & (3-3i)\beta & 3/8 \end{pmatrix}, \tag{B.1}$$

i.e., $a_{11}, a_{22}, b_{11}, b_{22}$, and two of the four free off-diagonal parameters have been fixed. The eigenvalues of this operator fall into the interval $0 \le \lambda_i \le 1\ \forall i$ iff $0 \le \alpha \le 0.3$ and $0 \le \beta \le 0.35$.

With these restrictions the corrected entropy $S_{corr}(\alpha,\beta)$ and the correlation function $\Delta(\alpha,\beta)$ can be determined as described above. For all possible values of α the graph of S_{corr} vs. Δ can be approximated to a high accuracy by the well-known ellipse equation. In Fig. 5 the approximated functions

$$S_{corr}(\Delta) = a\sqrt{1 - \frac{(\Delta - \Delta_{\min})^2}{b^2}} + S_{corr,\min} \tag{B.2}$$

are compared for a couple of representative values of α.

Appendix C

The expressions for the corrected entropy have the general form

$$S_{corr}(\varepsilon) = -(1-\varepsilon)\ln(1-\varepsilon) + c_1(1-\varepsilon^2). \tag{C.1}$$

We expand the logarithm into a power series and stop it after the second-order term.

$$\Rightarrow \quad S_{corr}(\varepsilon) = -\frac{\varepsilon^3}{2} - \left(\frac{1}{2} + c_1\right)\varepsilon^2 + \varepsilon + c_1. \tag{C.2}$$

Forming the square of $S_{corr}(\varepsilon)$ and omitting all terms with $\mathcal{O}(\varepsilon) > 2$ we obtain

$$S_{corr}(\varepsilon)^2 = -(2c_1^2 + c_1 - 1)\varepsilon^2 + 2c_1\varepsilon + c_1^2. \tag{C.3}$$

It has been shown in Sect. 4 that the correlation function $\Delta(\varepsilon)$ depends linearly on ε. We therefore approximate $\Delta(\varepsilon)$ by $c_2\varepsilon$, insert this into (C.3) and, after some manipulations, end up with

$$\frac{S_{corr}^2}{c_1^2} + \frac{(2c_1^2 + c_1 - 1)}{c_1^2 c_2^2}\Delta^2 = 1 + \frac{2}{c_1 c_2}\Delta \tag{C.4}$$

which, except for the perturbation linear in Δ, is the ellipse equation sought-after.

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